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A Unified Treatments of Vector Integration

by

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Introduction

In this paper we present a general approach to integration of functions where integrals and functions take their values in locally convex spaces. The integral is defined by use of a simple topological principle; the space of "simple" functions $\mathfrak{E}$ is a dense subspace of the space of integrable functions $\mathfrak{J}$ with respect to a suitable topology defined by upper norms and the integral—considered as a linear function on $\mathfrak{E}$—is extended continuously on $\mathfrak{J}$. Essentially this principle was already used by Stone [14] (for real-valued functions and integrals) and in the well known integration theory of Bourbaki [3], later by Brooks and Dinculeanu [4], Bichteler [2] and Schäfke [11] (in the Banach space case, respectively) and others.

In Chapter 1 we shortly describe the basis of a general integration theory (extension of the integral, convergence theorems); we omit the proofs which depend on well known arguments (s. [2], [11], [12]) as well as a further extension of the theory like a discussion of measurable sets and functions.

In Chapter 2 we construct upper norms suitable for the integral extension and we examine when upper norms were even upper gauges; upper gauges play a crucial role in connection with the validity of convergence theorems.

In Chapter 3 we show how the most important integrals can be obtained easily from the general integral of Chapter 1, even those integrals which originally were not defined by the principle of continuous extension mentioned above (for example, the two integrals of Dunford and Schwartz [6] and the integral of Kluvánek and Knowles [8]). Since we treat the locally convex case we are able to handle the weak integrals simultaneously.

The integral of this paper is applied in [15] to get a formulation of a more general version of the Riesz representation theorem without use of functionals.

Throughout this paper let $\mathbf{K}$ *denote the real field* $\mathbf{R}$ *or the complex field* $\mathbf{C}$, *let* E *and* F *be linear spaces over* $\mathbf{K}$ *and* Ω *a nonempty set.*

For two sets A and B let $\mathfrak{F}(A, B) := B^A$ denote the set of all mappings from A to B.

For a vector lattice $\mathfrak{E}'$ in $\mathfrak{F}(\Omega, \mathbf{R})$ a function $\| \quad \| : \mathfrak{E}' \to [0, \infty]$ is called an *upper norm* (s. [2]) if $\|0\| = 0$, $\|\alpha\varphi\| = |\alpha| \, \| \, |\varphi| \, \|$ for $\alpha \in \mathbf{K} \backslash \{0\}$, $\varphi \in \mathfrak{E}'$ (positive homogeneity), $\|\varphi\| \le \|\varphi_1\| + \|\varphi_2\|$ for $\varphi, \varphi_1, \varphi_2 \in \mathfrak{E}'$ with $|\varphi| \le |\varphi_1| + |\varphi_2|$ (subadditivity). The upper norms with values in $[0, \infty[$ are just the *Riesz seminorms* (s. [1, p. 38]).

1. A general integration theory

1.1. Basic assumptions, integral extension

In Chapter 1 of this paper we shall assume that

(A1) *F is a complete locally convex Hausdorff space whose topology is generated by a system of seminorms $(q_\nu)_{\nu\in\mathfrak{N}}$; E is a locally convex space and $(p_\nu)_{\nu\in\mathfrak{N}}$ a system of continuous seminorms on E; $\mathfrak{E}$ is a linear subspace of $\mathfrak{F}(\Omega, E)$ and i_0: $\mathfrak{E}\to F$ a linear map; to each $\nu\in\mathfrak{N}$ there corresponds an upper norm $\| \quad \|_\nu$: $\mathfrak{F}(\Omega, \mathbf{R})\to[0, \infty]$; for each $g\in\mathfrak{E}$, $\nu\in\mathfrak{N}$, the linear map i_0 satisfies $q_\nu(i_0(g))\le\|p_\nu\circ g\|_\nu<\infty$.*

$\mathfrak{F}:=\{f\in\mathfrak{F}(\Omega, E)\colon \forall\nu\in\mathfrak{N}\ \|p_\nu\circ f\|_\nu<\infty\}$ is a linear subspace of $\mathfrak{F}(\Omega, E)$ containing $\mathfrak{E}$. For $f\in\mathfrak{F}(\Omega, E)$, $\nu\in\mathfrak{N}$ we sometimes write $|||f|||_\nu$ instead of $\|p_\nu\circ f\|_\nu$. Now $(||| \quad |||_\nu)_{\nu\in\mathfrak{N}}$ is a system of seminorms on $\mathfrak{F}$ and $(\mathfrak{F}, (||| \quad |||_\nu)_{\nu\in\mathfrak{N}})$ is a locally convex space. i_0 is a continuous linear map with respect to this topology and therefore there exists a unique continuous linear extension i: $\mathfrak{I}\to F$ on the topological closure $\mathfrak{I}=\bar{\mathfrak{E}}$ of $\mathfrak{E}$ in $\mathfrak{F}$. $\mathfrak{I}$ is the space of *integrable functions* and i the *integral extension* of i_0.

We call a subset M of Ω a null set if for all $\nu\in\mathfrak{N}$ $\|\chi_M\|_\nu=0$ where χ_M denotes the characteristic function of M. We use the term "almost everywhere (a.e.)" corresponding to these null sets.

1.2. Lebesgue's dominated convergence theorem

Definition. Let $\mathfrak{E}'$ be a vector lattice in $\mathfrak{F}(\Omega, \mathbf{R})$ and $\| \quad \|$: $\mathfrak{E}'\to[0, \infty]$ an upper norm.

(a) $\| \quad \|$ is called a *weak upper gauge* if $\| \quad \|$ has values in $[0, \infty[$ and if for all nonnegative functions $\varphi, \varphi_1, \varphi_2, \dots\in\mathfrak{E}'$ $\sum_{n=1}^{\infty}\varphi_n\le\varphi$ implies $\|\varphi_n\|\to0$ $(n\to\infty)$.

(b) $\| \quad \|$ is called an *upper gauge* if $\| \quad \|$ has values in $[0, \infty[$ and if for all nonnegative functions $\varphi_1, \varphi_2, \varphi_3, \dots\in\mathfrak{E}'$ $\sup_n \|\sum_{k=1}^{n}\varphi_k\|<\infty$ implies $\|\varphi_n\|\to0$ $(n\to\infty)$.

Obviously any upper gauge is a weak upper gauge. The corresponding terms in [11] are "schwache Halbadditivität" and "Halbadditivität." Observe that we do not require a (weak) upper gauge to be an upper S-norm (s. [2, p. 73]) as in [2]. Weak upper gauges which are not upper S-norms are important in connection with the Riesz representation theorem [15].

In addition to the basic assumptions (A1) we now assume that

(A2) *to each $\nu \in \mathfrak{N}$ there corresponds a vector lattice $\mathfrak{E}'_\nu$ in $\mathfrak{F}(\Omega, \mathbf{R})$ containing $p_\nu \circ \mathfrak{E}$; for each $\nu \in \mathfrak{N}$ the upper norm $\|\ \ \|_\nu$ is σ-subadditive and the restriction $\|\ \ \|_\nu | \mathfrak{E}'_\nu$ is a weak upper gauge.*

Let $\mathfrak{I}'_\nu := \overline{\mathfrak{E}'_\nu}^{\|\ \ \|_\nu}$ denote the topological closure of $\mathfrak{E}'_\nu$ in the locally convex space $(\{\varphi \in \mathfrak{F}(\Omega, \mathbf{R})\colon \|\varphi\|_\nu < \infty\}, \|\ \ \|_\nu)$ $(\nu \in \mathfrak{N})$. In all important cases (s. (2.2.2)) a unique vector lattice $\mathfrak{E}'$ can be found in $\mathfrak{F}(\Omega, \mathbf{R})$ in such a way that $\mathfrak{E}'$ is independent of ν and for all $\nu \in \mathfrak{N}$ with $p_\nu \neq 0$ $p_\nu \circ \mathfrak{E}$ is the positive cone of $\mathfrak{E}'$.

THEOREM (1.2.1). *Suppose that* (A1) *and* (A2) *hold. Assume*
(1) *$(f_n) \in \mathfrak{I}^{\mathbf{N}}$, $f \in \mathfrak{F}(\Omega, E)$ and $f_n(x) \to f(x)$ a.e. and*
(2) *for each $\nu \in \mathfrak{N}$ there is a function $\varphi_\nu \in \mathfrak{I}'_\nu$ with $p_\nu \circ f_n \leq \varphi_\nu$ $(n \in \mathbf{N})$.*
Then for all $\nu \in \mathfrak{N}$ $|||f_n - f|||_\nu \to 0$ consequently $f \in \mathfrak{I}$ and $i(f_n) \to i(f)$ $(n \to \infty)$.

To prove the theorem, for each $\nu \in \mathfrak{N}$ we apply [12, Satz 3.4.2] to get $|||f_n - f|||_\nu \to 0$; here the topology induced by ρ^* in [12] corresponds to the topology induced by $|||\ \ |||_\nu$ for a fixed $\nu \in \mathfrak{N}$. The following theorem (as well as Theorem (1.2.1)) can be obtained by [16, (3.5.6)] using the statements of [16, p. 54].

THEOREM (1.2.2). *Suppose that* (A1) *and* (A2) *hold. If for all $\nu \in \mathfrak{N}$ $\|\ \ \|_\nu | \mathfrak{E}'_\nu$ is even an upper gauge then the assumption* (2) *of Theorem* (1.2.1) *can be weakened to*

$$(2)' \qquad \sup_n \|\sup_{k \leq n} p_\nu \circ f_k\|_\nu < \infty \qquad (\nu \in \mathfrak{N}).$$

(2)′ is satisfied if there are functions $\varphi_\nu \in \mathfrak{F}(\Omega, \mathbf{R})$ with $\|\varphi_\nu\|_\nu < \infty$ and $p_\nu \circ f_n \leq \varphi_\nu$ $(n \in \mathbf{N}, \nu \in \mathfrak{N})$.

If we can find a bounded set B in E with $f_n(\Omega) \subset B$ $(n \in \mathbf{N})$ we can choose $\varphi_\nu := \sup_{y \in B} p_\nu(y) \cdot \chi_\Omega$ as dominating functions in (1.2.1), (1.2.2); (2)′ is satisfied if $\|\chi_\Omega\|_\nu < \infty$ $\{\nu \in \mathfrak{N}\}$; and (2) if $\chi_\Omega \in \mathfrak{E}'_\nu$ $(\nu \in \mathfrak{N})$.

1.3. The space $(\mathfrak{F}, (|||\ \ |||_\nu)_{\nu \in \mathfrak{N}})$

In this section we assume in addition to the basic assumptions (A1) of 1.1

(A3)
1. *$\|\ \ \|_\nu$ is σ-subadditive for all $\nu \in \mathfrak{N}$;*
2. *E is a Fréchet space and the topology of E is generated by the system $(p_\nu)_{\nu \in \mathfrak{N}}$;*
3. *for all $\nu_1, \nu_2 \in \mathfrak{N}$ $p_{\nu_1} = p_{\nu_2}$ or for all $\nu_1, \nu_2 \in \mathfrak{N}$ $\|\ \ \|_{\nu_1} = \|\ \ \|_{\nu_2}$.*

The last assumption 3. could be weakened to: For each pair $\nu_1, \nu_2 \in \mathfrak{N}$ there is a $\nu_3 \in \mathfrak{N}$ with $p_{\nu_1} \leq p_{\nu_3}$ and $\|\ \ \|_{\nu_2} \leq \|\ \ \|_{\nu_3}$.

The proof of the following generalization of the theorem of Levi is obvious (compare [11, Satz 3.2.1]).

THEOREM (1.3.1). *Suppose that* (A1) *and* (A3) *hold. Let $(f_n) \in \mathfrak{F}^{\mathbf{N}}$ and*

$f \in \mathfrak{F}(\Omega, E)$ such that $f(x)=\sum_{n=1}^{\infty} f_n(x)$ for all those x that make the series convergent. For all $\nu \in \mathfrak{N}$ let $\| \sum_{k=n}^{\infty} p_\nu \circ f_k \|_\nu \to 0$ $(n \to \infty)$. Then we have $f(x)=\sum_{n=1}^{\infty} f_n(x)$ a.e., $|||f-\sum_{k=1}^{n} f_k|||_\nu \to 0$ $(\nu \in \mathfrak{N})$ and $f \in \mathfrak{F}$. Moreover, $(f_n) \in \mathfrak{J}^{\mathbf{N}}$ implies $f \in \mathfrak{J}$ and $i(f)=\sum_{n=1}^{\infty} i(f_n)$.

The assumption $\| \sum_{k=n}^{\infty} p_\nu \circ f_k \|_\nu \to 0$ $(n \to \infty)$ is satisfied if $\sum_{k=1}^{\infty} |||f_k|||_\nu < \infty$. Further conditions which imply this assumption are given in the next proposition.

PROPOSITION (1.3.2). *Suppose* (A1), (A2), (A3) *and let $(f_n) \in \mathfrak{J}^{\mathbf{N}}$. If one of the following two conditions is fulfilled*

(1) $$\sum_{k=1}^{\infty} p_\nu \circ f_k \leq \varphi_\nu \in \mathfrak{J}'_\nu \qquad (\nu \in \mathfrak{N}),$$

(2) $\| \quad \|_\nu \,|\, \mathfrak{E}'_\nu$ *is even an upper gauge and*

$$\sup_n \left\| \sum_{k=1}^{n} p_\nu \circ f_k \right\|_\nu < \infty \qquad (\nu \in \mathfrak{N}),$$

then we have

$$\left\| \sum_{k=n}^{\infty} p_\nu \circ f_k \right\|_\nu \to 0 \qquad (n \to \infty) \quad \textit{for all} \quad \nu \in \mathfrak{N}.$$

The proof is obtained by the same arguments as in Lemma 1.4.3 and Theorem 1.5.2. of [11], see also [16, (3.2.2)].

PROPOSITION (1.3.3). *Suppose that* (A1) *and* (A2) *hold. Then for $f \in \mathfrak{F}$ we have $f(x)=0$ a.e. iff for all $\nu \in \mathfrak{N}$ $|||f|||_\nu=0$ (compare* [12, (2.2.3)]).

THEOREM (1.3.4). *Suppose that* (A1) *and* (A3) *hold. If $\mathfrak{N}$ is at most countable then*

(a) *$(\mathfrak{F}, (|||\quad|||_\nu)_{\nu \in \mathfrak{N}})$ and $(\mathfrak{J}, (|||\quad|||_\nu)_{\nu \in \mathfrak{N}})$ are Fréchet spaces.*

(b) *For $(f_n) \in \mathfrak{F}^{\mathbf{N}}$ and $f \in \mathfrak{F}$ (f_n) converges to f in $(\mathfrak{F}, (|||\quad|||_\nu)_{\nu \in \mathfrak{N}})$ iff (f_n) is a Cauchy sequence in $(\mathfrak{F}, (|||\quad|||_\nu)_{\nu \in \mathfrak{N}})$ and there exists a subsequence (f_{γ_n}) of (f_n) such that $f_{\gamma_n}(x) \to f(x)$ a.e.*

The proof is obtained by the same arguments as in [11, Satz 2.5.2, Satz 2.4.2, Satz 1.2.5].

2. Construction of suitable upper norms

If we want to apply the general integration theory of Chapter 1 in concrete cases we have to construct a dominating family $(\|\quad\|_\nu)_{\nu \in \mathfrak{N}}$ of upper norms to a given "elementary integral" i_0 (s. (A1)). To obtain such upper norms we first consider the semivariation of i_0 (s. Section 2.2) on a suitable vector lattice in $\mathfrak{F}(\Omega, \mathbf{R})$ and then we extend this to an upper norm on $\mathfrak{F}(\Omega, \mathbf{R})$.

2.1. Extension of upper norms

In this section let $\| \quad \|_0$: $\mathfrak{E}' \to [0, \infty]$ be an upper norm on a vector lattice $\mathfrak{E}'$ in $\mathfrak{F}(\Omega, \mathbf{R})$, κ a cardinal different from 0 and $\| \quad \|_\kappa$: $\mathfrak{F}(\Omega, \mathbf{R}) \to [0, \infty]$ defined by $\|\varphi\|_\kappa = \inf\{\sup_{\psi\in\Psi}\|\psi\|_0$: Ψ is an increasingly directed set of (nonnegative) functions of $\mathfrak{E}'$ with a cardinality $|\Psi| \leq \kappa$ and $\sup \Psi \geq |\varphi|\}$.

The following theorem is well known.

THEOREM (2.1.1). (a) *$\| \quad \|_\kappa$ is an upper norm; for infinite cardinals κ $\| \quad \|_\kappa$ is σ-subadditive.*

(b) *For all $\psi \in \mathfrak{E}'$ we have $\|\psi\|_\kappa < \|\psi\|_0$.*

(c) *For $\varphi \in \mathfrak{E}'$ we have $\|\varphi\|_\kappa = \|\varphi\|_0$ iff for every increasingly directed set Ψ of (nonnegative) functions of $\mathfrak{E}'$ $|\Psi| \leq \kappa$ and* $\sup \Psi \geq |\varphi|$ *imply* $\sup_{\psi\in\Psi}\|\psi\|_0 \geq \|\varphi\|_0$.

The larger κ is the smaller $\| \quad \|_\kappa$ is; this is of interest because small upper norms yield coarse topologies and consequently large spaces of integrable functions.

For finite κ $\|\varphi\|_\kappa = \inf\{\|\psi\|_0 : |\varphi| \leq \psi \in \mathfrak{E}'\}$ $(\varphi \in \mathfrak{F}(\Omega, \mathbf{R}))$; in this case $\| \quad \|_\kappa$ is an extension of $\| \quad \|_0$ and is called the *Riemann extension* of $\| \quad \|_0$.

For countable κ $(=\sigma$: $|\mathbf{N}|)$ and $\varphi \in \mathfrak{F}(\Omega, \mathbf{R})$ $\|\varphi\|_\sigma = \inf\{\sup_n \|\psi_n\|_0$: $(\psi_n) \in \mathfrak{E}'^{\mathbf{N}}$, $0 \leq \psi_1 \leq \psi_2 \cdots$, $\sup_n \psi_n \geq |\varphi|\}$; $\| \quad \|_\sigma$ is called the *Stone* or *Lebesgue upper norm* associated with $\| \quad \|_0$.

For all cardinals $\kappa \geq |\mathfrak{E}'|$ and $\varphi \in \mathfrak{F}(\Omega, \mathbf{R})$ $\|\varphi\|_\kappa = \inf\{\sup_{\psi\in\Psi}\{\|\psi\|_0$: Ψ is an increasingly directed set of (nonnegative) functions of $\mathfrak{E}'$ with $\sup \Psi \geq |\varphi|\}$; for $\kappa \geq |\mathfrak{E}'|$ $\| \quad \|_\kappa$ is called the *Bourbaki upper norm* associated with $\| \quad \|_0$.

A sufficient condition for $\| \quad \|_\kappa$ to be an extension of $\| \quad \|_0$ is that $\| \quad \|_0$ is κ-smooth i.e. that for every decreasingly directed subset Ψ of $\mathfrak{E}'$ $\inf \Psi = 0$ and $|\Psi| \leq \kappa$ imply $\inf_{\psi\in\Psi} \|\psi\|_0 = 0$; see also (2.2.3).

2.2. The semivariation

In this section let

$\mathfrak{E}$ be a linear subspace of $\mathfrak{F}(\Omega, E)$,
i_0; $\mathfrak{E} \to F$ a linear map, p a seminorm on E
and q a seminorm on F.

Definition. For $\varphi \in \mathfrak{F}(\Omega, \mathbf{R})$

$$(p, q) - \|\varphi\|_{i_0} := \sup\{q(i_0(g)) : g \in \mathfrak{E},\ p \circ g \leq |\varphi|\}$$

is called the *(p, q)-semivariation* of φ with respect to i_0. In this section we abbreviate $(p, q) - \| \quad \|_{i_0}$ to $\| \quad \|_0$. Obviously $\| \quad \|_0$ has the following properties.

PROPOSITION (2.2.1). (a) *$q(i_0(g)) \leq \|p \circ g\|_0$ for $g \in \mathfrak{E}$.*

(b) *For an upper norm $\| \quad \|$: $\mathfrak{F}(\Omega, \mathbf{R}) \to [0, \infty]$ the followings are equivalent:*

(1) *For all $g \in \mathfrak{E}$ $q(i_0(g)) \leq \|p \circ g\|$.*

(2) *For all functions $\psi \in p \circ \mathfrak{E}$ $\|\psi\|_0 \leq \|\psi\|$.*

(3) $\| \quad \|_0 \leq \| \quad \|$.

(c) *For* $\varphi, \psi \in \mathfrak{F}(\Omega, \mathbf{R})$ $|\varphi| \leq |\psi|$ *implies* $\|\varphi\|_0 \leq \|\psi\|_0$.

(d) $\|\alpha\varphi\|_0 = |\alpha| \, \| \, |\varphi| \, \|_0$ *for all* $\varphi \in \mathfrak{F}(\Omega, \mathbf{R})$, $\alpha \in \mathbf{K} \setminus \{0\}$.

For a ring $\mathbf{A}$ in the power set $\mathbf{P}(\Omega)$ of Ω let

$$\mathfrak{E}(\mathbf{A}, E) := \left\{ \sum_{k=1}^{n} \chi_{A_k} y_k : n \in \mathbf{N};\ A_k \in \mathbf{A},\ y_k \in E \qquad (k=1, \cdots, n) \right\}.$$

If in this case $\|0\|_0 = 0$ we have for all $M \subset \Omega$

$$\|\chi_M\|_0 = \sup \left\{ q\left(\sum_{k=1}^{n} \mu(A_k) y_k \right) : n \in \mathbf{N};\ y_k \in E,\ p(y_k) \leq 1, \right.$$

$$\left. M \supset A_k \in \mathbf{A},\ A_k \text{ mutually disjoint } (k=1, \cdots, n) \right\},$$

where μ: $\mathbf{A} \to L(E, F) := \{T \in \mathfrak{F}(E, F) : T \text{ linear}\}$ denotes the content (=finitely additive set function) defined by $\mu(A)y = i_0(\chi_A y)$ $(A \in \mathbf{A},\ y \in E)$.

THEOREM (2.2.2). *Let E be a locally convex space and p a continuous seminorm on E. Let $\mathfrak{E}$ and $\mathfrak{E}'$ be one of the following pairs of subspaces of $\mathfrak{F}(\Omega, E)$ and $\mathfrak{F}(\Omega, \mathbf{R})$, respectively.*

1) $\mathfrak{E} = \mathfrak{E}(\mathbf{A}, E)$, $\mathfrak{E}' = \mathfrak{E}(\mathbf{A}, \mathbf{R})$ *where* $\mathbf{A}$ *is a ring in* $\mathbf{P}(\Omega)$;

2) *the spaces of all continuous maps (or of all continuous maps with bounded range or of all continuous maps with totally bounded range) in case that Ω is a topological space;*

3) *the spaces of all continuous maps with compact support in case that Ω is a locally compact Hausdorff space.*

Then $\|\varphi\|_0 < \infty$ for all $\varphi \in \mathfrak{E}'$ implies that $\| \quad \|_0 | \mathfrak{E}'$ is an upper norm.

Proof. To verify the subadditivity let $\varphi, \psi \in \mathfrak{E}'$ and $f \in \mathfrak{E}$ such that $p \circ f \leq |\psi + \varphi|$. For each $n \in \mathbf{N}$ the functions

$$g_n := \frac{\inf\{|\psi|, p \circ f\}}{\sup\{1/n, p \circ f\}} \cdot f, \qquad h_n := \frac{p \circ f}{\sup\{1/n, p \circ f\}} \cdot f - g_n,$$

$$f_n := f - g_n - h_n \text{ belong to } \mathfrak{E} \text{ and } p \circ g_n \leq |\psi|,$$

Hence

$$q(i_0(f)) \leq q(i_0(g_n)) + q(i_0(h_n)) + q(i_0(f_n)) \leq \|\psi\|_0 + \|\varphi\|_0 + \|\psi_n\|_0 \qquad (n \in \mathbf{N}).$$

With a suitable $\vartheta \in \mathfrak{E}'$ we have $\psi_n \leq (1/n)\vartheta$ $(n \in \mathbf{N})$ hence $\|\psi_n\|_0 \leq (1/n)\|\vartheta\|_0 \to 0$ $(n \to \infty)$. It follows that $q(i_0(f)) \leq \|\psi\|_0 + \|\varphi\|_0$ and finally $\|\psi + \varphi\|_0 \leq \|\psi\|_0 + \|\varphi\|_0$.

It is easy to carry over this proof to more general function spaces $\mathfrak{E}$ and $\mathfrak{E}'$ to show that $\| \quad \|_0 | \mathfrak{E}'$ is an upper norm.

THEOREM (2.2.3). *Let $\mathbf{A}$ be a ring in $\mathbf{P}(\Omega)$, $\mathfrak{E} = \mathfrak{E}(\mathbf{A}, E)$ and $\mathfrak{E}' = \mathfrak{E}(\mathbf{A}, \mathbf{R})$.*

Let $\|\varphi\|_0<\infty$ *for all* $\varphi\in\mathfrak{E}'$ *and suppose that the content* μ: $\mathbf{A}\to L(E, F)$ *defined by* $\mu(A)y := i_0(\chi_A y)$ *is a q-measure (i.e.* $q(\mu(\bigcup_{k=1}^{\infty} A_k)y-\sum_{k=1}^{n}\mu(A_k)y)\to 0$ $(n\to\infty)$ *for every sequence* (A_n) *of mutually disjoint sets of* $\mathbf{A}$ *such that* $\bigcup_{k=1}^{\infty}A_k\in\mathbf{A}$ *and every* $y\in E$). *Then the Lebesgue upper norm* $\|\ \ \|_\sigma$ *associated with* $\|\ \ \|_0\,|\,\mathfrak{E}'$ *is an extension of* $\|\ \ \|_0\,|\,\mathfrak{E}'$ (s, 2.1).

This theorem can be proved in a way similar to [11, 5.6.1.].

2.3. Locally s-bounded outer contents and weak upper gauges

In [11, Section 5.5] an important criterion for "schwache Halbadditivität" of a σ-subadditive upper norm is given. It is possible to give a much simpler proof of this result by use of Proposition (2.3.1)—even for upper norms which are not σ-subadditive. We use the more general criterion in [15].

Definition. A map μ from a ring of sets $\mathbf{A}$ with values in $[0, \infty]$ or in a topological vector space is called *(locally) s-bounded* if for all sequences $(A_n)\in\mathbf{A}^{\mathrm{N}}$ of mutually disjoint sets (such that $\bigcup_{n=1}^{\infty}A_n$ is contained in a $A\in\mathbf{A}$) $\mu(A_n)$ converges to 0.

PROPOSITION (2.3.1). *Let* $\mathbf{A}$ *be a ring in* $\mathbf{P}(\Omega)$ *and* μ^*: $\mathbf{A}\to[0, \infty]$ *an outer content (=finitely subadditive set function). Then the following statements are equivalent*

(1) μ^* *is s-bounded.*

(2) *For all sequences* $(A_n)\in\mathbf{A}^{\mathrm{N}}$ *with* $\sum_{n=1}^{\infty}\chi_{A_n}$ *bounded,* $(\mu^*(A_n))$ *converges to* 0.

Proof. (2)$\curvearrowright$(1) is trivial (1)$\curvearrowright$(2): We show by induction on $k\in\mathbf{N}$.

$$(*)\qquad \forall (A_n)\in\mathbf{A}^{\mathrm{N}}\qquad \sum_{n=1}^{\infty}\chi_{A_n}\le k \curvearrowright \mu^*(A_n)\to 0\qquad (n\to\infty)\,.$$

From (1) $(*)$ follows for $k=1$. Suppose $(*)$ is valid for a fixed $k\in\mathbf{N}$. To show that $(*)$ is valid also for $k+1$ let $(A_n)\in\mathbf{A}^{\mathrm{N}}$ and $\sum_{n=1}^{\infty}\chi_{A_n}\le k+1$. With $B_n := A_n\setminus\bigcup_{j=1}^{n-1}A_j\in\mathbf{A}$ and $C_n := A_n\cap\bigcup_{j=1}^{n-1}A_j\in\mathbf{A}$ we have $\sum_{n=1}^{\infty}\chi_{B_n}\le 1$ and $\sum_{n=1}^{\infty}\chi_{C_n}\le k$, therefore by the induction hypothesis $\lim\mu^*(B_n)=\lim\mu^*(C_n)=0$ and since $\mu^*(A_n)\le\mu^*(B_n)+\mu^*(C_n)$ also $\lim_n\mu^*(A_n)=0$.

THEOREM (2.3.2). *Let* $\mathbf{A}$ *be a ring in* $\mathbf{P}(\Omega)$, $\|\ \ \|$: $\mathfrak{E}(\mathbf{A}, \mathbf{R})\to[0, \infty]$ *an upper norm and* μ^*: $\mathbf{A}\to[0, \infty]$ *defined by* $\mu^*(A)=\|\chi_A\|$. *Then the following statements are equivalent*:

(1) μ^* *is a locally s-bounded finite outer content.*

(2) $\|\ \ \|$ *is a weak upper gauge.*

Proof. (2)$\curvearrowright$(1) is trivial. To show (1)$\curvearrowright$(2) let $\varphi_1, \varphi_2, \varphi_3, \cdots$ be nonnegative functions of $\mathfrak{E}(\mathbf{A}, \mathbf{R})$, $\alpha>0$, $A\in\mathbf{A}$ and $\sum_{n=1}^{\infty}\varphi_n\le\alpha\chi_A$. Further let $\varepsilon>0$ and for $n\in\mathbf{N}$ $A_n := \{x\in\Omega\colon \varphi_n(x)>\varepsilon/\beta\}$ $(\in\mathbf{A})$, where $\beta := \mu^*(A)+1$. Then we have $\varphi_n\le\alpha\chi_{A_n}+(\varepsilon/\beta)\chi_A$, $(\varepsilon/\beta)\chi_{A_n}\le\varphi_n$ $(n\in\mathbf{N})$ and $\sum_{n=1}^{\infty}\chi_{A_n}\le(\alpha\beta/\varepsilon)\cdot\chi_A$. By (2.3.1) (replace Ω by A)

there is a $n_0 \in \mathbf{N}$ such that $\mu^*(A_n) \leq \varepsilon/\alpha$ for all $n \geq n_0$. For $n \geq n_0$ it follows that $\|\varphi_n\| \leq \alpha\mu^*(A_n) + (\varepsilon/\beta)\mu^*(A) < 2\varepsilon$.

As a corollary of (2.3.2) we obtain the result of Schäfke [11, 5.5.2] mentioned above where we need not require $\| \quad \|$ to be σ-subadditive.

2.4. Locally s-bounded contents and weak upper gauges

In this section let

$\mathbf{A}$ be a ring in $\mathbf{P}(\Omega)$, $\mu\colon \mathbf{A} \to L(E, F)$ a content, $i_0\colon \mathfrak{E}(\mathbf{A}, E) \to F$ the linear map defined by $i_0(\chi_A y) = \mu(A)y$ ($A \in \mathbf{A}$, $y \in E$), p a norm on a finite dimensional vector space E, q a seminorm on F.

We set $\| \quad \|_0 := (p, q) - \| \quad \|_{i_0}$, $\mu^*(M) := \|\chi_M\|_0$ ($M \subset \Omega$), $|T| := \sup\{q(Ty)\colon y \in E,\ p(y) \leq 1\}$ $\{T \in L(E, F))$ and $\tilde{\mu}(M) := \sup\{|\mu(A)|\colon A \in \mathbf{A},\ A \subset M\}$ ($M \subset \Omega$). *Obviously* $\tilde{\mu}(M) \leq \mu^*(M)$ ($M \subset \Omega$).

LEMMA (2.4.1). *Let $z_1, \cdots, z_m \in F$ and $\alpha_1, \cdots, \alpha_m \in [0, 1]$. Then*

$$q\left(\sum_{j=1}^{m} \alpha_j z_j\right) \leq \sup\left\{q\left(\sum_{j\in J} z_j\right)\colon J \subset \{1, \cdots, m\}\right\}.$$

To prove this well know inequality suppose that $0 =: \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_m \leq 1$ and write

$$\sum_{j=1}^{m} \alpha_j z_j = \sum_{k=1}^{m} \left((a_k - \alpha_{k-1}) \sum_{j=k}^{m} z_j\right).$$

PROPOSITION (2.4.2) (s. [6, IV, 10.4. (b)]). *Let $n \in \mathbf{N}$, $E = \mathbf{K}^n$; $\max_{k=1}^{n} |\alpha_k| = p(\alpha_1, \cdots, \alpha_n)$ for $(\alpha_1, \cdots, \alpha_n) \in \mathbf{K}^n$. Then*

(a) *$\mu^*(M) \leq 4n\tilde{\mu}(M)$ for all $M \subset \Omega$.*

(b) *In case of $n=1$ $\|\,|f|\,\|_0 \leq 4 \cdot \sup\{q(i_0(\chi_A f))\colon A \in \mathbf{A}\}$ for all $f \in \mathfrak{E}(\mathbf{A}, \mathbf{K})$.*

Proof. (a) Let $M \subset \Omega$ and $g \in \mathfrak{E}(\mathbf{A}, \mathbf{K}^n)$ with $p \circ g \leq \chi_M$. Represent g as

$$g = \sum_{k=1}^{n} \sum_{j=1}^{m} \chi_{A_j} \alpha_{jk} e_k$$

where $A_1, \cdots, A_m \in \mathbf{A}$ are mutually disjoint subsets of M, $\alpha_{jk} \in \mathbf{K}$, $|\alpha_{jk}| \leq 1$ and $e_k = (\delta_{kl})_{l=1}^{n}$ is the k-th unit vector of $\mathbf{K}^n$. Using (2.4.1) we obtain

$$q\left(\sum_{j=1}^{m} \alpha_{jk}\mu(A_j)e_k\right) \leq 4\sup\left\{q\left(\sum_{j\in J} \mu(A_j)e_k\right)\colon J \subset \{1, \cdots, m\}\right\}$$

$$= 4\sup\left\{q\left(\mu\left(\bigcup_{j\in J} A_j\right)e_k\right)\colon J \subset \{1, \cdots, m\}\right\} \leq 4\tilde{\mu}(M),$$

hence

$$q(i_0(g)) \leq \sum_{k=1}^{n} q\left(\sum_{j=1}^{m} \alpha_{jk}\mu(A_j)e_k \right) \leq 4n\tilde{\mu}(M).$$

This proves (a).

(b) Let $n=1$, $f \in \mathfrak{E}(\mathbf{A}, \mathbf{K})$ and let i_f, μ_f be defined by

$$i_f(g) = i_0(gf),\ \mu_f(A) = i_f(\chi_A) = i_0(\chi_A f) \in F = L(\mathbf{K}, F)\ (g \in \mathfrak{E}(\mathbf{A}, \mathbf{K}),\ A \in \mathbf{A}).$$

Now let $g \in \mathfrak{E}(\mathbf{A}, \mathbf{K})$ with $|g| \leq |f|$ and define a function $h \in \mathfrak{E}(\mathbf{A}, \mathbf{K})$ by $h(x) := g(x)/f(x)$ if $f(x) \neq 0$ and $h(x)=0$ if $f(x)=0$. From $hf=g$ and $|h| \leq \chi_\Omega$ it follows that $q(i_0(g)) = q(i_f(h)) \leq (p, q) - \|\chi_\Omega\|_{i_f}$. Now (a) yields $q(i_0(g)) \leq 4\tilde{\mu}_f(\Omega) = 4 \cdot \sup\{q(i_0(\chi_A f)) \colon A \in \mathbf{A}\}$. This proves (b).

For locally convex spaces X, Y let $\mathfrak{L}(X, Y)$ be the set of all continuous linear maps from X to Y equipped with the topology of uniform convergence on bounded sets.

THEOREM (2.4.3). *The following statements are equivalent:*

(1) $\mu \colon \mathbf{A} \to \mathfrak{L}((E, p), (F, q))$ *is locally s-bounded.*

(2) *For each* $y \in E$ *the map* $\mathbf{A} \ni A \mapsto q(\mu(A)y)$ *is locally s-bounded.*

(3) $\| \quad \|_0 \,|\, \mathfrak{E}(\mathbf{A}, \mathbf{R})$ *is a weak upper gauge.*

Proof. (1) $\Leftrightarrow$ (2): For a basis $e_1, \cdots, e_n$ of E there exists a constant $\alpha \in [0, \infty[$ such that for $A \in \mathbf{A}$ and $y \in E$ with $p(y) \leq 1$ we have

$$q(\mu(A)y) \leq |\mu(A)| \leq \alpha \cdot \sum_{k=1}^{n} q(\mu(A)e_k).$$

This proves (1) $\Leftrightarrow$ (2).

(1) $\Leftrightarrow$ (3): Obviously (1) is equivalent to

(1)′ $\tilde{\mu} \,|\, \mathbf{A}$ is locally s-bounded;

in view of (2.3.2) (3) is equivalent to

(3)′ $\mu^* \,|\, \mathbf{A}$ is locally s-bounded and finite.

By (2.4.2) there exists a $\beta \in [0, \infty[$ such that $\tilde{\mu} \leq \mu^* \leq \beta\tilde{\mu}$; further it is well known that (1)′ implies $\tilde{\mu}(A) < \infty (A \in \mathbf{A})$ (s. [5, p. 9]). This proves (1)′ $\Leftrightarrow$ (3)′ and therefore (1) $\Leftrightarrow$ (3).

COROLLARY (2.4.4). *Let* $\mathbf{A}$ *be a* δ*-ring and* μ *a q-measure. Then* $\| \quad \|_0 \,|\, \mathfrak{E}(\mathbf{A}, \mathbf{R})$ *is a weak upper gauge.*

Since in this cace $A \mapsto q(\mu(A)y)$ is locally s-bounded this follows from (2.4.3).

COROLLARY (2.4.5). *Let* $\omega \colon \mathbf{A} \to [0, \infty[$ *be a content and* μ ω*-continuous* (*i.e.* $\forall \varepsilon > 0\ \exists \delta > 0\ \forall A \in \mathbf{A}\ \omega(A) < \delta \Rightarrow |\mu(A)| < \varepsilon$). *Then* $\| \quad \|_0 \,|\, \mathfrak{E}(\mathbf{A}, \mathbf{R})$ *is a weak upper gauge.*

Since ω is locally s-bounded the same is true for $A \mapsto |\mu(A)|$. Thus (2.4.5) again follows from (2.4.3).

As an example for (2.4.5) consider a Hilbert space $(F, \langle , \rangle)$ with the usual norm q,

$q(z) := \langle z, z\rangle^{1/2}$ and an orthogonal content μ: $\mathbf{A} \to F\,(=L(\mathbf{K}, F))$ (i.e. $\langle \mu(A), \mu(B)\rangle = 0$ for disjoint $A, B \in \mathbf{A}$); here for ω we can choose $\omega(A) := \langle \mu(A), \mu(A)\rangle$.

2.5. Supplement

In this section we want to demonstrate how the several steps of Chapters 1 and 2 can be combined to get the integral extension and how, starting from an elementary integral, the remaining assumptions of (A1) can be fulfilled.

In the following let

> *E be a locally convex space; F a complete locally convex Hausdorff space whose topology is generated by the system of seminorms $(q_\nu)_{\nu \in \mathfrak{N}}$; $\mathfrak{E}$ and $\mathfrak{E}'$ as in* (2.2.2); *i_0: $\mathfrak{E} \to F$ a linear map.*

i_0 is called *of finite semivariation* (*κ-smooth* for a cardinal κ) if for each continuous seminorm q on F there exists a continuous seminorm p on E such that $(p, q) - \|\ \ \|_{i_0} | \mathfrak{E}'$ is finite (κ-smooth).

Obviously κ-smoothness for infinite κ implies finite semivariation.

Remark. Let P be a set of continuous seminorms on E such that for every continuous seminorm p_0 on E there is a $p \in P$ and $n \in \mathbf{N}$ with $p_0 \le n \cdot p$. Then i_0 is of finite semivariation (κ-smooth) iff for each $\nu \in \mathfrak{N}$ there is a $p \in P$ such that $(p, q_\nu) - \|\ \ \|_{i_0} | \mathfrak{E}'$ is finite (κ-smooth).

We now assume that i_0 is of finite semivariation and κ-smooth for a fixed κ; in case $\kappa = |\mathbf{N}|$, $\mathfrak{E} = \mathfrak{E}(\mathbf{A}, E)$ the latter assumption may be replaced by requiring that the content μ: $\mathbf{A} \to L(E, F)$ belonging to i_0 is a q_ν-measure for each $\nu \in \mathfrak{N}$.

Then for each $\nu \in \mathfrak{N}$ we can choose a continuous seminorm p_ν on E such that the semivariation $(p_\nu, q_\nu) - \|\ \ \|_{i_0} | \mathfrak{E}'$ is a finite upper norm which can be extended to an upper norm $\|\ \ \|_\nu$: $\mathfrak{F}(\Omega, \mathbf{R}) \to [0, \infty]$, $\|\varphi\|_\nu := \inf\{\sup_{\psi \in \Psi} (p_\nu, q_\nu) - \|\psi\|_{i_0}$: Ψ is an increasingly directed set of nonnegative functions of $\mathfrak{E}'$, $|\Psi| \le \kappa$, $\sup \Psi \ge |\varphi|\}$ (s. (2.2.2), (2.2.3), 2.1). Now the assumptions (A1) are fulfilled and $\mathfrak{J}$ and i can be defined as in 2.1. If we assume in addition that κ is infinite and $(p_\nu, q_\nu) - \|\ \ \|_{i_0} | \mathfrak{E}'$ is a weak upper gauge for all $\nu \in \mathfrak{N}$, the assumptions (A2) are satisfied with $\mathfrak{E}'_\nu = \mathfrak{E}'$ ($\nu \in \mathfrak{N}$) and Lebesgue's theorem (1.2.1) is valid.

3. Special integrals

3.1. Integration of vector valued functions with respect to scalar measures

> *Let E and F be locally convex Hausdorff spaces. Assume that F is complete, and that E is an algebraical subspace of F such that the inclusion $E \hookrightarrow F$ is continuous.*
> *Let $\mathbf{A}$ be a ring in $\mathbf{P}(\Omega)$ and μ: $\mathbf{A} \to \mathbf{K}$ a measure of finite total variation v: $\mathbf{A} \to [0, \infty[$.*

Further let $(q_\nu)_{\nu\in\mathfrak{N}}$ be a system of seminorms generating the topology of F; $p_\nu := q_\nu | E \neq 0$ $(\nu\in\mathfrak{N})$; $\mathfrak{E} := \mathfrak{E}(\mathbf{A}, E)$, $\mathfrak{E}'_\nu := \mathfrak{E}' := \mathfrak{E}(\mathbf{A}, \mathbf{R})$ $(\nu\in\mathfrak{N})$, and i_0: $\mathfrak{E}\to F$ the linear map defined by $i_0(\chi_A y)=\mu(A)y$ $(A\in\mathbf{A},\ y\in E)$. For each $\nu\in\mathfrak{N}$ let $\|\ \ \|_\nu$ be the Lebesgue upper norm $\|\ \ \|_L$: $\mathfrak{F}(\Omega, \mathbf{R})\to[0, \infty]$ defined by

$$\|\varphi\|_L=\inf\{\sup \textstyle\int\psi_n\, d\nu\colon (\psi_n)\in\mathfrak{E}'^{\mathfrak{N}},\ 0\leq\psi_1\leq\psi_2\leq\cdots,\ \sup\psi_n\geq|\varphi|\}$$

Then the assumptions (A1) are fulfilled and $\mathfrak{F}$, $\mathfrak{J}$ and i can be defined as in 1.1. Since $\int|\psi|d\nu=(p_\nu, q_\nu)-\|\psi\|_{i_0}$ $(\psi\in\mathfrak{E}',\ \nu\in\mathfrak{N})$ this integral is a special case of the integral introduced in Section 2.5. As usual we write $i(f)=\int f d\mu$ for $f\in\mathfrak{J}$. Observe that the topology of $(\mathfrak{F}, (|||\ \ |||_\nu)_{\nu\in\mathfrak{N}})$ and consequently $\mathfrak{J}$ and i depend only on the topology of F and not on the special choice of $(q_\nu)_{\nu\in\mathfrak{N}}$.

$\|\ \ \|_L$ is σ-subadditive and $\|\ \ \|_L|\mathfrak{E}'$ is an upper gauge since

$$\|\varphi+\psi\|_L=\textstyle\int(\varphi+\psi)d\nu=\int\varphi d\nu+\int\psi d\nu=\|\varphi\|_L+\|\psi\|_L$$

for non-negative φ, $\psi\in\mathfrak{E}'$. Consequently the assumptions (A2) are satisfied and the convergence theorems of Section 1.2 hold. Moreover, if E is a Fréchet space and a topological subspace of F then the assumptions (A3) are also fulfilled.

In the following we specialize the integral of this section to obtain well known integrals.

(3.1.1). Let $E=F$ be the same Banach space. We choose $\mathfrak{N}=\{\nu_0\}$ and for p_{ν_0} and q_{ν_0} the norm of the given space.

Then the integral of this section yields the *Bochner integral.* This follows easily from the usual definition of the Bochner integral (s. e.g. [5, p. 44]) in view of (1.3.4) (b). The Bochner integral coincides with the first integral of Dunford and Schwartz [6, p. 112] for measures. Schäfke [13] has shown that the first integral of Dunford and Schwartz for contents can also be obtained in a similar manner using "upper norms" which are not necessarily positive homogen. It should be noted that the positive homogeneity of upper norms is not essential in Chapter 1 (s. [11], [12]).

(3.1.2). Let $F=E'^*$ be the algebraical dual of the topological dual of E equipped with the weak*-topology $\sigma(E'^*, E')$. Then F is the completion of $(E, \sigma(E, E'))$. Let $\mathfrak{N}$ be a basis of E' and $q_\xi(z)=|z(\xi)|$ $(\xi\in\mathfrak{N},\ z\in F)$. We write more precisely $\mathfrak{J}(E)$ instead of $\mathfrak{J}$ and $\mathfrak{J}(\mathbf{K})$ in case that $E=\mathbf{K}$.

THEOREM. (1) *For $f\in\mathfrak{J}(E)$ and $\xi\in E'$ we have $\xi\circ f\in\mathfrak{J}(\mathbf{K})$ and $(\int f d\mu)(\xi)=\int\xi\circ f d\mu$.*

(2) $\mathfrak{J}(E)=\{f\in\mathfrak{F}(\Omega, E)\colon \forall\xi\in E'\ \xi\circ f\in\mathfrak{J}(\mathbf{K})\}$.

Proof. (1) and hence one inclusion of (2) are easy consequences of the definitions. Now let $f\in\mathfrak{F}(\Omega, E)$ and $\xi\circ f\in\mathfrak{J}(\mathbf{K})$ for all $\xi\in E'$. To show that $f\in\mathfrak{J}(E)$ we construct a $g\in\mathfrak{E}$ with $|||f-g|||_{\xi_k}<\varepsilon$ $(k=1, \cdots, n)$ for a given $\varepsilon>0$ and distinct $\xi_1, \cdots, \xi_n\in\mathfrak{N}$. First there are $\psi_k\in\mathfrak{E}(\mathbf{A}, \mathbf{K})$ with $\|\xi_k\circ f-\psi_k\|_L<\varepsilon$. Since the $\xi_1, \cdots, \xi_n$ are linearly independent there are $y_1, \cdots, y_n\in E$ with $\xi_k(y_j)=\delta_{kj}$ $(j, k=1, \cdots, n)$.

Now $g := \sum_{k=1}^{n} \psi_k y_k \in \mathfrak{E}$ and $\xi_k \circ g = \psi_k$ hence $|||f-g|||_{\xi_k} = \|\xi_k \circ f - \psi_k\|_L < \varepsilon$ ($k = 1, \cdots, n$).

In view of this theorem $\mathfrak{J}(E)$ is just the space of *weakly μ-integrable* functions also called the space of *Dunford-integrable* functions and $\mathfrak{J}_p(E) := \{f \in \mathfrak{J}(E)$: $\forall A \in \mathbf{A} \int \chi_A f d\mu \in E\}$ the space of *Pettis-integrable* functions (s. [5, p. 52]).

3.2. Integration of scalar valued functions with respect to vector valued measures

Let F be a complete locally convex Hausforff space, $\mathbf{A}$ a ring in $\mathbf{P}(\Omega)$ and μ: $\mathbf{A} \to F$ a measure of finite semivariation (the latter means in view of (2.4.2) (a) that $\sup\{q(\mu(B))$: $A \supset B \in \mathbf{A}\} < \infty$ for every $A \in \mathbf{A}$ and every continuous seminorm q on F).

Further let $(q_\nu)_{\nu \in \mathfrak{N}}$ be a system of seminorms generating the topology of F; for all $\nu \in \mathfrak{N}$ let p_ν be the absolute value on $E := \mathbf{K}$; $\mathfrak{E} = \mathfrak{E}(\mathbf{A}, \mathbf{K})$, $\mathfrak{E}' = \mathfrak{E}(\mathbf{A}, \mathbf{R})$, i_0: $\mathfrak{E} \to F$ the linear map defined by $i_0(\chi_A) := \mu(A)$ and $\| \quad \|_\nu$ the Lebesgue upper norm associated with the semivariation $(p_\nu, q_\nu) - \| \quad \|_{i_0} | \mathfrak{E}'$ ($\nu \in \mathfrak{N}$). Now the assumptions (A1) and (A3) are satisfied and $\mathfrak{F}$, $\mathfrak{J}$, i can be defined as in 1.1. This is again a special case of the integral of 2.5. As usual we write $\int_A f d\mu = i(f\chi_A)$ for $f \in \mathfrak{J}$, $A \in \mathbf{A}$. The topology of $(\mathfrak{F}, (||| \quad |||_\nu)_{\nu \in \mathfrak{N}})$ and consequently $\mathfrak{J}$ and i depend only on the topology of F and not on the special choice of $(q_\nu)_{\nu \in \mathfrak{N}}$.

If $\mathbf{A}$ is a δ-ring then every F-valued measure is of finite semivariation and $\| \quad \|_\nu | \mathfrak{E}'$ is a weak upper gauge for all $\nu \in \mathfrak{N}$ (s. (2.4.4)), therefore with $\mathfrak{E}'_\nu := \mathfrak{E}'$ the assumptions (A2) are also satisfied in this case.

In the following we get some well known integrals as special cases of the integral defined above.

(3.2.1) First we prove

THEOREM. *Let $\mathbf{A}$ be a σ-algebra and (f_n) a sequence in $\mathfrak{E}$ such that $f_n(x)$ converges in $\mathbf{K}$ a.e. Then the following statements are equivalent*

(1) *$(\int_A f_n d\mu)_n$ converges in F for all $A \in \mathbf{A}$.*

(2) *(f_n) is a Cauchy sequence in $(\mathfrak{F}, (||| \quad |||_\nu)_{\nu \in \mathfrak{N}})$.*

Proof. (2) $\curvearrowright$ (1) follows from

$$q_\nu\left(\int_A f_n d\mu - \int_A f_m d\mu\right) \leq |||(f_n - f_m)\chi_A|||_\nu \leq |||f_n - f_m|||_\nu .$$

(1) $\curvearrowright$ (2): We may assume that $f_n(x)$ converges for all $x \in \Omega$. We set $B_k := \{x \in \Omega$: $\forall n \in \mathbf{N}\, |f_n(x)| \leq k\}$ ($k \in \mathbf{N}$) then $\mathbf{A} \ni A_k := \Omega \backslash B_k \downarrow \varnothing$. We apply the theorem of Vitali-Hahn-Saks ([9, Theorem 4], [10, Theorem 5]) to the measures $\mathbf{A} \ni A \mapsto \int_A f_n d\mu$. Therefore for every $\nu \in \mathfrak{N}$ $\sup\{q_\nu(\int_B f_n d\mu)$: $\mathbf{A} \ni B \subset A_k\}$ and hence, by (2.4.2) (b), $|||f_n \chi_{A_k}|||_\nu$ converges to 0 ($k \to \infty$) uniformly in $n \in \mathbf{N}$. By Lebesgue's

Theorem (1.2.1) $(f_n\chi_{B_k})_n$ is a Cauchy sequence in $(\mathfrak{F}, \|\|\ \ \|\|_v)_{v\in\mathfrak{N}})$ for all $k\in\mathbf{N}$. Now from $\|\|f_n-f_m\|\|_v\le\|\|(f_n-f_m)\chi_{B_k}\|\|_v+\|\|f_n\chi_{A_k}\|\|_v+\|\|f_m\chi_{A_k}\|\|_v$ the assertion follows.

By virtue of (1.3.4) (b) we obtain

COROLLARY. *Let* $\mathbf{A}$ *be a* σ*-algebra and* F *a Fréchet space. Then a function* $f\in\mathfrak{F}(\Omega, \mathbf{K})$ *belongs to* $\mathfrak{I}$ *iff there exists a sequence* (f_n) *in* $\mathfrak{E}$ *such that*

(i) $f_n(x)\to f(x)$ *a.e. and*

(ii) $\int_A f_n d\mu$ *converges in* F *for all* $A\in\mathbf{A}$.

In this case $\int_A f d\mu=\lim_{n\to\infty}\int_A f_n d\mu$.

Hence for a σ-algebra $\mathbf{A}$ and a Banach space F the integral of this section is just the *second integral of Dunford and Schwartz* [6, p. 323] (observe that the null sets of Dunford and Schwartz are the same as ours).

(3.2.2). Let F_0 be a locally convex Hausforff space and $F=F_0'^*$ the algebraical dual of the topological dual of F_0 equipped with the weak*-topology $\sigma(F_0'^*, F_0')$. Further let μ: $\mathbf{A}\to F_0$ be a measure of finite semivariation; μ can also be considered as a measure μ: $\mathbf{A}\to F$. We choose $\mathfrak{N}=F_0'$ and $q_\xi(z)=|z(\xi)|$ $(z\in F)$ for $\xi\in F_0'$. Then for $\xi\in F_0'$ $\|\ \ \|_\xi$ is just the Lebesgue upper norm associated with the scalar measure $\xi\circ\mu$ (s. 3.1). Let us call the space $\mathfrak{I}$ of integrable functions more precisely $\mathfrak{I}(\mu)$. Then $\mathfrak{I}(\xi\circ\mu)$ is the closure of $\mathfrak{E}$ in $(\{f\in\mathfrak{F}(\infty, \mathbf{K})\colon \|\|f\|\|_\xi<\infty\}, \|\|\ \ \|\|_\xi)$.

THEOREM. (1) *For all* $f\in\mathfrak{I}(\mu)$ *and* $\xi\in F_0'$ *we have* $f\in\mathfrak{I}(\xi\circ\mu)$ *and* $(\int f d\mu)(\xi)=\int f d(\xi\circ\mu)$.

(2) $\mathfrak{I}(\mu)=\bigcap_{\xi\in F_0'}\mathfrak{I}(\xi\circ\mu)$.

Proof. (1) and thus one inclusion in (2) follow immediately from the remarks above. Now let $f\in\mathfrak{I}(\xi\circ\mu)$ for all $\xi\in F_0'$. To show that $f\in\mathfrak{I}(\mu)$ we construct a $g\in\mathfrak{E}$ with $\|\|f-g\|\|_{\xi_k}<\varepsilon$ $(k=1, \cdots, n)$ for a given $\varepsilon>0$ and $\xi_1, \cdots, \xi_n\in F_0'$. Since f is integrable with respect to the scalar measures $\xi_k\circ\mu$ f is also integrable with respect to the total variations v_k: $\mathbf{A}\to[0, \infty[$ of these measures and therefore it is integrable with respect to $v=\sum_{k=1}^n v_k$ (s. e.g. [7, Satz 4.1]). Hence there is a $g\in\mathfrak{E}$ with $\|\|f-g\|\|_L<\varepsilon$ where $\|\ \ \|_L$ denotes the Lebesgue upper norm for v. It then follows $\|\|f-g\|\|_{\xi_k}\le\|\|f-g\|\|_L<\varepsilon$ $(k=1, \cdots, n)$.

Hence the integral of (3.2.2) is the usual *weak integral with respect to* μ.

(3.2.3). Let the assumptions of this section be given and let $\mathbf{A}$ be a σ-algebra. Then $f\in\mathfrak{F}(\Omega, \mathbf{K})$ is integrable *in the sense of Kluvánek and Knowles* ([8, p. 21] with respect to μ if f is $\mathbf{A}$-measurable, weak integrable in the sense of (3.2.2) (replace the space F_0 of (3.2.2) by the given space F) and if the weak integral of f over any set $A\in\mathbf{A}$ belongs to F.

New we can prove

THEOREM. *For every* $f\in\mathfrak{F}(\Omega, \mathbf{K})$ *the followings are equivalent*:

(1) $f\in\mathfrak{I}$ *and* f *is* $\mathbf{A}$*-measurable*;

(2) *f is integrable in the sense of Kluvánek and Knowles. In this case $i(f)$ is equal to the weak integral of f.*

Proof. Obviously every $f \in \mathfrak{I}$ is weak integrable and $i(f)$ is equal to the weak integral of f. Therefore we only have to show that every function f which is integrable in the sense of Kluvánek and Knowles belongs to $\mathfrak{I}$. We may assume that f is nonnegative. Since f is **A**-measurable there is a sequence $0 \leq f_1 \leq f_2 \leq \cdots$ in $\mathfrak{E}'$ with $\sup_n f_n = f$. The theorem of Lebesgue for the integral of Kluvánek and Knowles [8, Theorem 2, p. 30] implies $f_n \to f$ in $(\mathfrak{F}, (|||\ \ |||_\nu)_{\nu \in \mathfrak{N}})$ thus $f \in \mathfrak{I}$. Notice [8, Lemma 2(3), p. 23] and $\|\varphi\|_\nu = (p_\nu, q_\nu) - \|\varphi\|_{i_0}$ ($\varphi \in \mathfrak{F}(\Omega, \mathbf{R})$, φ **A**-measurable, $\nu \in \mathfrak{N}$).

Using (1.3.4) (b) we obtain

COROLLARY. *Let F be a Fréchet space and let the null sets (s. 1.1) be contained in* **A**. *Then $\mathfrak{I}$ consists exactly of the functions integrable in the sense of Kluvánek and Knowles.*

3.3. The integral of Brooks and Dinculeanu [4]

> *Let E and F be Banach spaces, p and q the norms on E and F respectively;*
> **A** *a δ-ring in* $\mathbf{P}(\Omega)$, *μ: $\mathbf{A} \to \mathfrak{L}(E, F)$ a measure of finite semivariation (i.e. the elementary integral i_0: $\mathfrak{E}(\mathbf{A}, E) \to F$ belonging to μ is of finite semivariation in the sense of Section 2.5).*

To define an integral extension of i_0 Brooks and Dinculeanu first consider for each $\xi \in F'$ the measure μ_ξ: $\mathbf{A} \to E'$, $\mu_\xi(A) := \xi \circ \mu(A)$ and the total variation $|\mu_\xi|$: $\mathbf{A} \to [0, \infty[$. The set $N := \{|\mu_\xi|: \xi \in F', |\xi| \leq 1\}$ of positive measures induces an upper norm $N(\cdot)$ (s. [4, p. 352]). We can obtain this upper norm by using the results of Section 2.1 in the following way. For each $\rho \in N$ let $\|\ \ \|_\rho$ be the Lebesgue upper norm associated with $\psi \mapsto \int |\psi| d\rho$ ($\psi \in \mathfrak{E}' := \mathfrak{E}(\mathbf{A}, \mathbf{R})$). For $\varphi \in \mathfrak{F}(\Omega, \mathbf{R})$, $f \in \mathfrak{F}(\Omega, E)$ let $\|\varphi\| := \sup\{\|\varphi\chi_A\|_\rho: A \in \mathbf{A}, \rho \in N\}$ and $|||f||| := \|p \circ f\|$; then we have $N(\varphi) = \|\varphi\|$ and $N(f) = |||f|||$ whenever $N(\varphi)$ and $N(f)$ are defined in [4]. Now $\|\ \ \|$ is a σ-subadditive upper norm which extends the semivariation $(p, q) - \|\ \ \|_{i_0} | \mathfrak{E}'$. Therefore the assumptions (A1) are fulfilled and $\mathfrak{I}$ and i can be defined as in 1.1. Brooks and Dinculeanu define the integral similarly (using $N(\cdot)$ in place of $\|\ \ \|$). Thus it is easy to see that with the notations of [4, p. 353 and p. 360] we have $\mathfrak{I} = L^1_E(N)$ and $i(f) = \int f d\mu$ for $f \in \mathfrak{I}$. Moreover, the N-null sets of [4, p. 352] are just the null sets of Section 1.1.

If we replace the upper norm $\|\ \ \|$ by the Lebesgue upper norm $\|\ \ \|_L$ associated with the semivariation $(p, q) - \|\ \ \|_{i_0} | \mathfrak{E}'$ as in Section 2.5 we obtain the same integration theory apart from an eventually smaller class of null sets. If we assume that all sets $B \subset \Omega$ with $\|\chi_B\| = 0$ belong to **A** the two upper norms $\|\ \ \|$ and $\|\ \ \|_L$ yield the same spaces $\mathfrak{I}$ of integrable functions and the same integral i. This can be proved with (1.3.4) since $\|\psi\| = \|\psi\|_L$ ($\psi \in \mathfrak{E}'$).

In [4, Section 3] it is assumed additionally that N is a relatively weakly compact subset of ca ($\mathbf{A}$), the space of all real-valued measures on $\mathbf{A}$ equipped with the topology generated by the system of seminorms $(q_A)_{A\in\mathbf{A}}$, $q_A(\mu) := |\mu|(A)$ (=total variation of μ on A). This just means that $\| \quad \| \,|\, \mathfrak{E}'$ is a weak upper gauge. Indeed, the following statements are equivalent:

(1) N is relatively weakly compact in ca ($\mathbf{A}$).
(2) The measures $\rho\in N$ are uniformly σ-additive.
(3) $\mathbf{A}\ni A\mapsto \|\chi_A\|$ is order continuous.
(4) $\mathbf{A}\ni A\mapsto \|\chi_A\|$ is locally s-bounded.
(5) $\| \quad \| \,|\, \mathfrak{E}'$ is a weak upper gauge.

The equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) are proved in [4, p. 357]. Since $\mathbf{A}$ is a δ-ring we have (3) $\Leftrightarrow$ (4) and by (2.3.2) (4) $\Leftrightarrow$ (5).

In [4, Section 8] the Beppo-Levi-property of N is studied. N possesses the Beppo-Levi-property iff $\| \quad \| \,|\, \mathfrak{E}'$ is an upper gauge.

In several sections of the paper of Brooks and Dinculeanu [4] Lebesgue-type spaces are studied without presence of a measure μ. The set N is allowed to be any nonvoid bounded subset of ca ($\mathbf{A}$). Evidently such a discussion would also be possible in our theory.

3.4. The integrals of Bourbaki [3]

Let Ω be a locally compact Hausdorff space; F a complete locally convex Hausdorff space, whose topology is generated by the system of seminorms $(q_\nu)_{\nu\in\mathfrak{N}}$; E a locally convex Hausdorff space and $(p_\nu)_{\nu\in\mathfrak{N}}$ a system of continuous seminorms on E.

Let $\mathfrak{E}=\mathfrak{K}(\Omega, E)$ and $\mathfrak{E}'=\mathfrak{K}(\Omega, \mathbf{R})$ be the sets of all continuous functions with compact support from Ω to E and $\mathbf{R}$, respectively; i_0: $\mathfrak{E}\to F$ a linear map such that for all $\nu\in\mathfrak{N}$ and all compact subsets K of Ω $(p_\nu, q_\nu)-\|\chi_K\|_{i_0}<\infty$.

Then by (2.2.2) and Dini's theorem $(p_\nu, q_\nu)-\| \quad \|_{i_0} \,|\, \mathfrak{E}'$ is a τ-smooth upper norm ($\tau := |\mathfrak{E}|$ for all $\nu\in\mathfrak{N}$. Hence for all $\nu\in\mathfrak{N}$ the Bourbaki upper norm $\| \quad \|_\nu$ associated with $(p_\nu, q_\nu)-\| \quad \|_{i_0} \,|\, \mathfrak{E}'$ (s. Section 2.1) extends $(p_\nu, q_\nu)-\| \quad \|_{i_0} \,|\, \mathfrak{E}'$. Now the assumptions (A1) are satisfied and $\mathfrak{J}$ and i can be defined as in 1.1. This integral is a special case of the integral introduced in Section 2.5. In the following we use in place of the upper norms $\| \quad \|_\nu$ also the upper norms $\| \quad \|_{\nu,l} := \sup\{\|\varphi\chi_K\|_\nu$: K a compact subset of $\Omega\}$ ($\varphi\in\mathfrak{F}(\Omega, \mathbf{R})$, $\nu\in\mathfrak{N}$) to include local null sets.

The integral of this section generalizes the integrals of Bourbaki:

(3.4.1). As in (3.1.1) let $E=F$ be the same Banach space, $\mathfrak{N}=\{\nu_0\}$ and $p_{\nu_0}=q_{\nu_0}$ the norm of the given Banach space.

Let μ: $\mathfrak{K}(\Omega, \mathbf{K})\to\mathbf{K}$ be a measure in the sense of Bourbaki [3, III, §1, Def. 2] and i_0: $\mathfrak{K}(\Omega, E)\to E$ the elementary integral belonging to μ [3, III, §3]. With the notations of [3, III, §1.6 and IV, §1 and V, §1] we have $\|\varphi\|_{\nu_0}=\int^*|\varphi|\,d|\mu|$ and $\|\varphi\|_{\nu_0,l}=\int^{\bullet}|\varphi|\,d|\mu|$ ($\varphi\in\mathfrak{F}(\Omega, \mathbf{R})$). Therefore the integral of this section yields the integral of

[3, IV, §4] and, replacing $\| \quad \|_{v_0}$ by $\| \quad \|_{v_0, l}$, the integral of [3, V, §1.3].

(3.4.2). Now let as in (3.1.2) $F = E'^*$ equipped with the $\sigma(E'^*, E')$-topology. Let $\mathfrak{N}$ be a basis of E', $q_\xi(z) = |z(\xi)|$ $(\xi \in \mathfrak{N}, z \in F)$ and $p_\xi = q_\xi | E$. Let μ: $\mathfrak{K}(\Omega, \mathbf{K}) \to \mathbf{K}$ be a measure in the sense of Bourbaki and i_0: $\mathfrak{K}(\Omega, E) \to F$ the elementary integral defined by $(i_0(g)(\xi) := \mu(\xi \circ g)$. Then we have $\|\varphi\|_{\xi, l} = \int^{\bullet} |\varphi| \, d|\mu|$ for all $\xi \in \mathfrak{N}$ and $\varphi \in \mathfrak{F}(\Omega, \mathbf{R})$. The same argument as in (3.1.2) shows that the integral of this section using the upper norms $\| \quad \|_{\xi, l}$ yields the integral of [3, VI, §1].

(3.4.3). Now let as in (3.2.2) $E = \mathbf{K}$, F_0 a locally convex Hausdorff space, $F = F_0'^*$ equipped with the $\sigma(F_0'^*, F_0')$-topology, $\mathfrak{N} = F_0'$ and for $\xi \in F_0'$ p_ξ the absolute value on $\mathbf{K}$ and $q_\xi(z) = |z\{\xi\}|$ $(z \in F)$. Further let m: $\mathfrak{K}(\Omega, \mathbf{K}) \to F_0$ be a vector measure in the sense of Bourbaki [3, VI, §2, Def. 1] and $i_0 = m$. Then we have $\|\varphi\|_{\xi, l} = \int^{\bullet} |\varphi| \, d|\xi \circ m|$ for all $\xi \in F_0'$ and $\varphi \in \mathfrak{F}(\Omega, \mathbf{R})$. The same argument as in (3.2.2) (compare the proof of [3, VI, §2, Prop. 1] shows that the integral of this section using the upper norms $\| \quad \|_{\xi, l}$ yields the integral of [3, VI, §2].

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